Math 206 Sample Final Exam Problems

Question 1 Determine the solution set to the following system of equations:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & & & & & & & = & 2 \\ -x_1 & & & + & x_3 & + & 3x_4 & + & 4x_5 & = & 1 \\ & & x_2 & + & x_3 & + & 4x_4 & + & 4x_5 & = & 0 \\ x_1 & + & x_2 & & & + & x_4 & & & = & -1 \end{array}$$

Question 2

Consider the coefficient matrix from problem 1, $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 4 & 4 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$.

a) Find a basis for the kernel of A.
b) The kernel of A is a ____–dimensional subspace of ____–dimensional Euclidean space.
c) Find a basis for the image of A.
d) The image of A is a ____–dimensional subspace of ____–dimensional Euclidean space.

Question 3 Let A be an $n \times n$ matrix. If A is invertible can you conclude that...

a) A is row equivalent to the identity matrix?	**YES**	**NO**
b) the rows of A are linearly independent?	**YES**	**NO**
c) A is a diagonalizable matrix?	**YES**	**NO**
d) 0 is not an eigenvalue of A?	**YES**	**NO**
e) the system $A\mathbf{x} = 0$ has infinitely many solutions?	**YES**	**NO**
f) the determinant of A is non–zero?	**YES**	**NO**
g) the system $A\mathbf{x} = \mathbf{b}$ has a solution for all $n \times 1$ matrices $\mathbf{b}$?	**YES**	**NO**

Question 4 Let $A = \begin{bmatrix} 2 & 2 \\ 7 & -3 \end{bmatrix}$.

a) Compute the eigenvalues of A.
b) Compute the associated eigenvectors of A.
c) Find a matrix P so that $P^{-1}AP$ is a diagonal matrix.
d) Use your answer to part c) to compute A^7.

Question 5 Let W denote the subspace of $\mathbb{R}^3$ spanned by the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

a) What is the dimension of W? Explain.
b) Find a basis for W.

Question 6 Let A be a 5×5 matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 2$, $\lambda_4 = 3$, $\lambda_5 = 5$. Compute $\det(A^2 - 9I)$. Explain your answer. Hint: What is $\det(A - 3I)$?

Question 7 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3, x_1 + x_3, 2x_1 + x_2 + 3x_3).$$

a) Find the standard matrix for the linear transformation T.
b) Find a basis for the image of T.

Question 8 Consider the basis $S = \{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ where $\vec{v}_1 = (1, -1)$, and $\vec{v}_2 = (2, 0)$. Let T be a linear operator from $\mathbb{R}^2$ to $\mathbb{R}^4$ such that $T(\vec{v}_1) = (1, 0, 1, 0)$, and $T(\vec{v}_2) = (0, 0, 2, 1)$. Find the general formula for $T(\vec{x})$.

Question 9 Let $W \subset \mathbb{R}^4$ be the subspace spanned by the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 4 \end{bmatrix}$,and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 4 \end{bmatrix}$. Find an orthonormal basis for W. What is the QR factorization of $[\vec{v}_1\, \vec{v}_2\, \vec{v}_3]$? Find the projection of the vector $\vec{u} = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$ onto the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$. Find the 3-dimensional volume of the parallelepiped formed by $\vec{v}_1\, \vec{v}_2, \vec{v}_3$.

Question 10 Compute the inverse and determinant of a 4×4 matrix.